

CONVERGENCE OF A CRYSTALLINE ALGORITHM FOR THE MOTION OF A SIMPLE CLOSED CONVEX CURVE BY WEIGHTED CURVATURE

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ABSTRACT. Motion by weighted mean curvature is a geometric evolution law for surfaces and represents steepest descent with respect to anisotropic surface energy. It has been proposed that this motion could be computed numerically by using a “crystalline” approximation to the surface energy in the evolution law. In this paper we prove the convergence of this numerical method for the case of simple closed convex curves in the plane.

1. INTRODUCTION

In the modeling of phase transitions it is often of interest to consider surface-energy-driven motion of interfaces. If the surface energy is anisotropic this leads to motion by weighted mean curvature (see Angenent and Gurtin [1]). Some materials have “crystalline” energies. The idea of taking a crystalline approximation of a convex surface-energy leads naturally to a numerical scheme for determining motion by weighted curvature. For a detailed explanation as well as for the physical and mathematical context of this work the reader is referred to the introduction and appendix of [3]. (Familiarity with the theory of surface-energy-driven motion of phase boundaries is *not* assumed either in this work or in [3]). The idea of taking a crystalline approximation to the surface-energy has been studied in the static case by Sullivan in [6] to find an area minimizing oriented hypersurface that spans a given boundary in space. The first attempt to study this idea in the dynamic context was [3], where we examined the case in which the interface is the graph of a function of one variable. The main result of [3] was the convergence of a certain numerical scheme for a quasilinear parabolic differential equation with constant Dirichlet or Neumann boundary conditions. We proved convergence in H^1 with a specified rate. Our method was somewhat similar to the convergence analysis for a Galerkin approximation.

Here we study the crystalline approximation of motion by weighted curvature for smooth simple closed convex curves. For motion by weighted curvature the normal velocity of the curve is its weighted curvature; the weighted curvature ϖ is

$$\varpi \triangleq (f + f'') \kappa,$$

if f (the interfacial energy) is a smooth function which depends only on the angle between the normal to the curve and a fixed axis and κ is the curvature. We assume throughout that both f and $f + f''$ are positive functions. We approximate the interface by a polygon with N (≥ 4) sides moving by weighted curvature (as defined below). All the interior angles of the polygon are equal and the sides of the polygon have fixed directions: the interior normal to the i th side is $N_i = -(\cos i\Delta\theta, \sin i\Delta\theta)$, where $\Delta\theta = 2\pi/N$, in a fixed coordinate system.

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It is standard to parametrize a convex curve by the angle θ between its normal and a fixed coordinate axis (here θ is the angle between the exterior normal and the x -axis). Using this parametrization the evolution equation for the weighted curvature of the interface is

$$\varpi_t = (f + f'')^{-1}(\varpi^2 \varpi_{\theta\theta} + \varpi^3)$$

(see Eq. (2.23) of Angenent and Gurtin [1]). On the other hand, the evolution equation for the weighted curvature ω_i of the i th side of the polygon turns out to be

$$\dot{\omega}_i = \left[f_i + \frac{f_{i+1} - 2f_i + f_{i-1}}{2(1 - \cos \Delta\theta)} \right]^{-1} \left[\omega_i^2 \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{2(1 - \cos \Delta\theta)} + \omega_i^3 \right]$$

($f_i \triangleq f(i\Delta\theta)$). To prove convergence of the crystalline approximation we shall derive the discrete analogue of estimates due to Gage and Hamilton [2] for the curvature and then use standard arguments to estimate $|\varpi(i\Delta\theta, t) - \omega_i(t)|$. This will enable us to estimate the distance between the curve and the polygon.

We mention that if the energy is isotropic, the initial curve is a circle, and the initial polygon is regular and circumscribed to the initial circle then the polygon remains regular and circumscribed to the circle as they collapse to a point. For other computational examples we refer the reader to Roosen and Taylor [5].

The organization of this paper is as follows: in Section 2 we set up the problem and give estimates for the weighted curvature; then in Section 3 we prove the convergence of the crystalline algorithm for motion by weighted curvature.

In this paper we consider only convex curves in the plane. A natural next step would be to consider general (i.e. nonconvex) simple closed curves. The paper by Grayson [4] might be relevant because it generalizes the results of [2] for convex curves to general embedded plane curves.

2. SETUP AND ESTIMATES FOR THE WEIGHTED CURVATURE

Consider a smooth simple closed curve in the plane. We say that the curve is moving by weighted curvature when its normal velocity equals its weighted curvature:

$$V(\theta, t) = \varpi(\theta, t) \triangleq [f(\theta) + f''(\theta)] \kappa(\theta, t) \quad (1)$$

as in Eq. (4.11) of Angenent and Gurtin [1]. (This is Eq. (41) of [3].) Here θ is the angle between the exterior normal and a fixed axis, κ is the curvature, and ϖ is the weighted curvature. We consider the case where f (the interfacial energy per unit length) is positive, C^3 , and such that $f + f'' > 0$. We are interested in studying convergence of an approximation scheme for this equation when the curve is convex. In this scheme we substitute the curve by a convex polygon with N sides, $N \geq 4$. The angle between two adjacent sides of the polygon is $\pi - \Delta\theta$, where

$$\Delta\theta = \frac{2\pi}{N}$$

is fixed, $0 < \Delta\theta \leq \pi/2$. The interior normals can be written $N_i \triangleq -(\cos i\Delta\theta, \sin i\Delta\theta)$ in a coordinate system that will henceforth remain fixed. We take θ to be the angle between the exterior normal and the x -axis. Whenever we refer to “sides” we mean the sides of this polygon. The i th side is the one with interior normal N_i and joins the i th vertex with the $(i+1)$ th one. We define $T_i \triangleq (-\sin i\Delta\theta, \cos i\Delta\theta)$ (see Fig. 1).

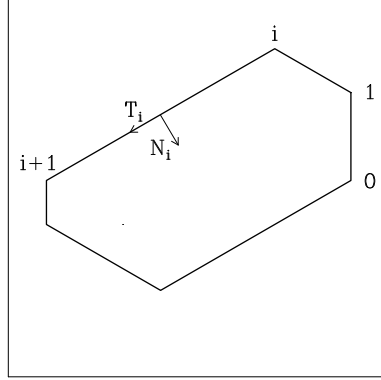


FIG. 1. A polygon with six sides and labeled corners.

Each side has zero curvature, of course, if curvature is defined in the standard way, namely the rate of change with respect to arc length of the angle between the tangent and a fixed axis; in this standard sense the curvature is not defined at the vertices of the polygon. However, curvature can alternatively be defined as the negative of the gradient of arclength; weighted curvature is the negative of the gradient of the interfacial energy. In this sense each side of the polygon does have nonzero (weighted) curvature. We denote by L_i , V_i , k_i , and ω_i the length, velocity, curvature, and weighted curvature, respectively, of the i th side. The normal velocity of the i th side in the approximation scheme is

$$V_i = \omega_i \quad (2)$$

in the direction N_i . Let $f_i \triangleq f(i\Delta\theta)$. As long as no sides disappear the weighted curvature of the i th side is

$$\omega_i \triangleq \left[f_i + \frac{f_{i+1} - 2f_i + f_{i-1}}{2(1 - \cos \Delta\theta)} \right] k_i \quad (3)$$

and its curvature is

$$k_i \triangleq \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \frac{1}{L_i} = 2 \tan \frac{\Delta\theta}{2} \frac{1}{L_i} \quad (4)$$

(see Eqs. (45) of [3]). Also,

$$\dot{L}_i = 2\omega_i \cot \Delta\theta - \omega_{i-1} \csc \Delta\theta - \omega_{i+1} \csc \Delta\theta \quad (5)$$

(see Eqs. (10.18) of [1]).

Define

$$\begin{aligned} g &\triangleq f + f'', & g_i &\triangleq f_i + \frac{f_{i+1} - 2f_i + f_{i-1}}{2(1 - \cos \Delta\theta)}, \\ h &\triangleq \frac{1}{g}, & \text{and } h_i &\triangleq \frac{1}{g_i}. \end{aligned}$$

We note that $g_i > 0$ for all $\Delta\theta$, $0 < \Delta\theta \leq \pi/2$, and all i . This is a consequence of the facts that (i) $g_i \leq 0$ is equivalent to $\Delta\theta \neq \pi/2$ and $f_i \geq (f_{i+1} + f_{i-1})/(2\cos \Delta\theta)$, (ii) the line through $-N_{i-1}/f_{i-1}$ and $-N_{i+1}/f_{i+1}$ intersects the line with direction $-N_i$ at $2\cos \Delta\theta / (f_{i+1} + f_{i-1}) \times (-N_i)$, and (iii) the curvature of $(\cos(\cdot), \sin(\cdot))/f(\cdot)$ is $f^3(f + f'')/\sqrt{(f^2 + f'^2)^3}$ and hence is positive, i.e. the polar diagram of $1/f$ (the Frank diagram) is convex. Therefore g_{\min} is bounded away from zero; g_{\max} is also bounded and these bounds are uniform in $\Delta\theta$. We use

the notation $(\cdot)_{\min}$, $(\cdot)_{\max}$, and $|\cdot|_{\max}$ for $\min_{0 \leq i \leq N-1} (\cdot)_i$, $\max_{0 \leq i \leq N-1} (\cdot)_i$, and $\max_{0 \leq i \leq N-1} |(\cdot)_i|$.

Next we compute the rate of change of the length of the polygon and of the area enclosed by it, and we compute the evolution equation for the weighted curvature of the sides. These computations remain valid as long as no side disappears. We could prove directly from Eqs. (3), (4), and (5) that this only happens when the polygon shrinks to a point or to a line (see [3] or Taylor [7]). Instead we deduce it later from an estimate for the maximum of the weighted curvature.

The rate of change of the (total) length of the polygon is

$$\begin{aligned}
 \dot{L} &= \sum_i \dot{L}_i \\
 &= \sum_i (2\omega_i \cot \Delta\theta - \omega_{i-1} \csc \Delta\theta - \omega_{i+1} \csc \Delta\theta) \\
 &= \sum_i 2(\cot \Delta\theta - \csc \Delta\theta) \omega_i \\
 &= -\sum_i 4g_i \frac{(\cot \Delta\theta - \csc \Delta\theta)^2}{L_i}.
 \end{aligned} \tag{6}$$

Throughout this paper we write \sum_i for $\sum_{i=0}^{N-1}$. The last equation shows that, as in the continuous case, the total length of the polygon is a decreasing function.

The area enclosed by the polygon can be computed by

$$A_{\Delta\theta} = \frac{1}{2} \sum_i d_i L_i$$

if we put the origin inside the polygon and let d_i be the distance between the origin and the line containing the i th side. The rate of change of the area is

$$\begin{aligned}
 \dot{A}_{\Delta\theta} &= \frac{1}{2} \sum_i \dot{d}_i L_i + \frac{1}{2} \sum_i d_i \dot{L}_i \\
 &= -\frac{1}{2} \sum_i \omega_i L_i + \frac{1}{2} \sum_i d_i (2\omega_i \cot \Delta\theta - \omega_{i-1} \csc \Delta\theta - \omega_{i+1} \csc \Delta\theta) \\
 &= -\frac{1}{2} \sum_i \omega_i L_i + \frac{1}{2} \sum_i \omega_i (2d_i \cot \Delta\theta - d_{i+1} \csc \Delta\theta - d_{i-1} \csc \Delta\theta) \\
 &= -\sum_i \omega_i L_i \\
 &= -\sum_i g_i \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \longrightarrow -\int_0^{2\pi} g(\theta) d\theta
 \end{aligned}$$

as $\Delta\theta \rightarrow 0$, since $\dot{d}_i = -\omega_i$ and $d_{i+1} \csc \Delta\theta + d_{i-1} \csc \Delta\theta - 2d_i \cot \Delta\theta = L_i$ (see Fig. 2).

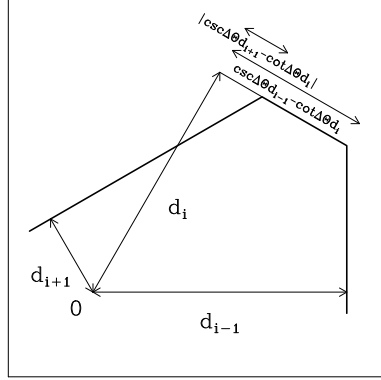


FIG. 2. The i th and part of the $(i-1)$ th and $(i+1)$ th sides. Here $N = 6$ and $i = 1$.

By Eq. (2.23) of Angenent and Gurtin [1]

$$\varpi_t = h(\varpi^2 \varpi_{\theta\theta} + \varpi^3). \quad (7)$$

We compute the discrete analogue of Eq. (7) using Eqs. (3), (4), and (5):

$$\begin{aligned} \dot{\omega}_i &= -g_i \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \frac{\dot{L}_i}{L_i^2} \\ &= -h_i \frac{\sin \Delta\theta}{2(1 - \cos \Delta\theta)} \omega_i^2 (2\omega_i \cot \Delta\theta - \omega_{i-1} \csc \Delta\theta - \omega_{i+1} \csc \Delta\theta) \\ &= h_i \left[\omega_i^2 \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{2(1 - \cos \Delta\theta)} + \omega_i^3 \right]. \end{aligned} \quad (8)$$

This is a system of ordinary differential equations determining the evolution of the ω_i 's.

Note that one can reconstruct the polygon at time t from the knowledge of the polygon at time zero and the ω_i 's. In fact the (normal) velocity of the line containing the i th side of the polygon is ω_i . Alternatively, it is easy to check that the velocity of the midpoint of the i th side is $\omega_i N_i - \frac{\omega_{i+1} - \omega_{i-1}}{2 \sin \Delta\theta} T_i$.

Later we shall need the following discrete version of Poincaré's inequality: Let p_0, \dots, p_M be $M+1$ real numbers with $p_0 = p_M = 0$. Then

$$\sum_{m=0}^{M-1} p_m^2 \leq \frac{1}{2(1 - \cos \frac{\pi}{M})} \sum_{m=0}^{M-1} (p_{m+1} - p_m)^2. \quad (9)$$

This is a restatement of the well know fact that the smallest eigenvalue of the $(M-1) \times (M-1)$ matrix

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

is $2(1 - \cos \frac{\pi}{M})$. We remark that we will use the fact that the denominator in Eq. (8) is $2(1 - \cos \Delta\theta)$ and not $(\Delta\theta)^2$ because inequality (9) is not valid if we substitute $2(1 - \cos \frac{\pi}{M})$ by $\frac{\pi^2}{M^2}$.

We now derive the discrete analogue of several estimates by Gage and Hamilton [2] for the curvature. The computations below are only valid as long as no side disappears.

First, note that Eqs. (8) show that $\omega_{\min}(t)$ is a nondecreasing function.

Second, we can bound the “ H^1 norm” of the “sequence ω_i ” in terms of its “ L^2 norm.” We compute

$$\begin{aligned} \frac{d}{dt} \sum_i \left[\omega_i^2 - \frac{(\omega_{i+1} - \omega_i)^2}{2(1 - \cos \Delta\theta)} \right] \Delta\theta &= 2 \sum_i \left[\omega_i \dot{\omega}_i - \frac{(\omega_{i+1} - \omega_i)(\dot{\omega}_{i+1} - \dot{\omega}_i)}{2(1 - \cos \Delta\theta)} \right] \Delta\theta \\ &= 2 \sum_i \dot{\omega}_i \left[\omega_i + \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{2(1 - \cos \Delta\theta)} \right] \Delta\theta \\ &= 2 \sum_i h_i \omega_i^2 \left[\omega_i + \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{2(1 - \cos \Delta\theta)} \right]^2 \Delta\theta. \end{aligned}$$

This is nonnegative so

$$\sum_i \frac{(\omega_{i+1} - \omega_i)^2}{2(1 - \cos \Delta\theta)} \Delta\theta \leq \sum_i \omega_i^2 \Delta\theta + c_1. \quad (10)$$

Here c_1 does not depend on time but only on the polygon at time zero. Note that if $\omega_{\max}(0)$ and $(\omega_{i+1}(0) - \omega_i(0))^2/[2(1 - \cos \Delta\theta)]$ are uniformly bounded then c_1 is also uniformly bounded. We may suppose $c_1 \geq 0$.

Third, if $A_{\Delta\theta} \geq \epsilon > 0$ then we can bound

$$\omega_* \triangleq \begin{cases} \max_{0 \leq j \leq N-1} \min_{j+1 \leq i \leq j+N/2} \omega_i & \text{for } N \text{ even,} \\ \max_{0 \leq j \leq N-1} \min_{j+1 \leq i \leq j+(N-1)/2} \omega_i & \text{for } N \text{ odd,} \end{cases}$$

which Gage and Hamilton call the “median weighted curvature.” Let j_0 be a value of j for which the maximum is assumed. A polygon with median weighted curvature ω_* lies between parallel lines whose distance is less than

$$\begin{aligned} \sum_{j=j_0}^{j_0+N/2} \sin((j-j_0)\Delta\theta) L_j &= \sum_{j=1}^{N/2} \sin(j\Delta\theta) g_{j_0+j} \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \frac{1}{\omega_{j_0+j}} \\ &\leq \frac{\sin \Delta\theta}{1 - \cos \Delta\theta} g_{\max} \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \frac{1}{\omega_*} \\ &= 2 \frac{g_{\max}}{\omega_*} \end{aligned}$$

for N even,

$$\begin{aligned}
\sum_{j=j_0}^{j_0+(N-1)/2} \sin((j-j_0)\Delta\theta) L_j &= \sum_{j=1}^{(N-1)/2} \sin(j\Delta\theta) g_{j_0+j} \frac{2(1-\cos\Delta\theta)}{\sin\Delta\theta} \frac{1}{\omega_{j_0+j}} \\
&\leq \frac{\sin\frac{\Delta\theta}{2}(\cos\frac{\Delta\theta}{2}+1)}{1-\cos\Delta\theta} g_{\max} \frac{2(1-\cos\Delta\theta)}{\sin\Delta\theta} \frac{1}{\omega_*} \\
&= (1 + \sec\frac{\Delta\theta}{2}) \frac{g_{\max}}{\omega_*} \\
&\leq (1 + \sec\frac{2\pi}{10}) \frac{g_{\max}}{\omega_*} \leq \frac{5}{2} \frac{g_{\max}}{\omega_*}
\end{aligned}$$

for N odd, and has diameter bounded by $L/2$. So

$$A_{\Delta\theta} \leq \frac{5}{2} \frac{g_{\max}}{\omega_*} \frac{L}{2} = \frac{5}{4} g_{\max} \frac{L}{\omega_*},$$

or

$$\omega_* \leq \frac{5}{4} g_{\max} \frac{L}{A_{\Delta\theta}}.$$

Note that if $A_{\Delta\theta}$ is bounded away from zero uniformly in $\Delta\theta$ then ω_* is also uniformly bounded.

Fourth, if ω_* is bounded then $\sum_i g_i \log \omega_i \Delta\theta$ is bounded. By Eqs. (8),

$$\begin{aligned}
\frac{d}{dt} \sum_i g_i \log \omega_i &= \sum_i \left[\omega_i \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{2(1-\cos\Delta\theta)} + \omega_i^2 \right] \\
&= \sum_i \left[\omega_i^2 - \frac{(\omega_{i+1} - \omega_i)^2}{2(1-\cos\Delta\theta)} \right].
\end{aligned}$$

Define $\tilde{I} \triangleq \{i \in \mathbf{N} | \omega_i > \omega_*\}$, divide \tilde{I} in maximal subsets of the form $\tilde{I}_j = \{i_j, i_j+1, \dots, i_j+M_j-2\}$, let $I_j \triangleq \{i_j-1, i_j, i_j+1, \dots, i_j+M_j-2\}$, $I \triangleq \cup_j I_j$, and $L \triangleq \mathbf{N} \setminus I$. Note that I_j has M_j elements and note also that $M_j-1 \leq N/2-1$, or $M_j \leq N/2$, because by the definition of ω_* there are at most $N/2-1$ ω_i 's for N even, $(N-1)/2-1$ ω_i 's for N odd, corresponding to adjacent sides and with $\omega_i > \omega_*$. We have

$$\sum_{i \in L \cap \{0, \dots, N-1\}} \left[\omega_i^2 - \frac{(\omega_{i+1} - \omega_i)^2}{2(1-\cos\Delta\theta)} \right] \leq \sum_{i \in L \cap \{0, \dots, N-1\}} \omega_*^2$$

and

$$\begin{aligned}
\sum_{i \in I_j} \left[\omega_i^2 - \frac{(\omega_{i+1} - \omega_i)^2}{2(1 - \cos \Delta\theta)} \right] &\leq \left[\omega_*^2 - \frac{(\omega_{i_j} - \omega_*)^2}{2(1 - \cos \Delta\theta)} \right] \\
&\quad + \sum_{i=i_j}^{i_j+M_j-3} \left[\omega_i^2 - \frac{(\omega_{i+1} - \omega_i)^2}{2(1 - \cos \Delta\theta)} \right] \\
&\quad + \left[\omega_{i_j+M_j-2}^2 - \frac{(\omega_* - \omega_{i_j+M_j-2})^2}{2(1 - \cos \Delta\theta)} \right] \\
&\leq 2\omega_* \sum_{i=i_j}^{i_j+M_j-2} \omega_i.
\end{aligned}$$

For the last inequality we have used Eq. (9) with $M = M_j$, $p_m = \omega_{i_j-1+m} - \omega_*$ for $1 \leq m \leq M_j - 1$, the fact that

$$\frac{1}{2(1 - \cos \frac{\pi}{M_j})} \leq \frac{1}{2(1 - \cos \frac{2\pi}{N})} = \frac{1}{2(1 - \cos \Delta\theta)},$$

and also that $M_j \geq 2$. The last estimates together with Eq. (6) imply

$$\frac{d}{dt} \sum_i g_i \log \omega_i \Delta\theta \leq \omega_*^2 \frac{2\pi}{\Delta\theta} \Delta\theta - 2\omega_* \frac{\sin \Delta\theta}{2(1 - \cos \Delta\theta)} \Delta\theta \sum_i \dot{L}_i.$$

Thus if $\omega_*(\tau) \leq \Omega$ for $0 \leq \tau \leq t$ then

$$\sum_i g_i \log \omega_i(t) \Delta\theta \leq \sum_i g_i \log \omega_i(0) \Delta\theta + 2\pi\Omega^2 t + 2\Omega[L(0) - L(t)].$$

Note that if $\omega_{\max}(0)$ and Ω are uniformly bounded then this bound is also uniform.

Fifth, if $\sum_i g_i \log \omega_i \Delta\theta$ is bounded then for any $\delta > 0$ we can find a constant $c_2 > 1$ such that $\omega_i \leq c_2$ except for $i\Delta\theta$ in intervals whose total length is less than δ . In fact, $\omega_i \geq c_2$ for L values of i and $L\Delta\theta \geq \delta$ implies

$$\begin{aligned}
\sum_i g_i \log \omega_i \Delta\theta &\geq g_{\min} \Delta\theta L \log c_2 + g_{\max} (N - L) \Delta\theta \log \omega_{\min} \\
&\geq g_{\min} \delta \log c_2 + g_{\max} (2\pi - \delta) \log \omega_{\min},
\end{aligned} \tag{11}$$

and this gives a contradiction when c_2 is large (we have assumed $\omega_{\min} < 1$). Here the constant c_2 depends on $\omega_{\min}(0)$. Note that if $\omega_{\min}(0)$ is bounded away from zero uniformly in $\Delta\theta$ then the bound for c_2 is also uniform.

Sixth and last, if $\omega_i \leq c_2$ except for $i\Delta\theta$ in intervals whose total length is less than δ and δ is small enough then ω_{\max} is bounded. In fact, $\omega_m \leq c_2$ and $n - m \leq \delta/\Delta\theta$

imply

$$\begin{aligned}
\omega_n &= \omega_m + \sum_{i=m}^{n-1} (\omega_{i+1} - \omega_i) \\
&\leq c_2 + \left[\sum_{i=m}^{n-1} \frac{2(1 - \cos \Delta\theta)}{\Delta\theta} \right]^{1/2} \left[\sum_{i=m}^{n-1} \frac{(\omega_{i+1} - \omega_i)^2}{2(1 - \cos \Delta\theta)} \Delta\theta \right]^{1/2} \\
&\leq c_2 + \sqrt{\delta} \frac{\sqrt{2(1 - \cos \Delta\theta)}}{\Delta\theta} \left[\sum_{i=m}^{n-1} \frac{(\omega_{i+1} - \omega_i)^2}{2(1 - \cos \Delta\theta)} \Delta\theta \right]^{1/2} \\
&\leq c_2 + \sqrt{\delta} \left[\sum_i \omega_i^2 \Delta\theta + c_1 \right]^{1/2}.
\end{aligned}$$

Hence

$$\omega_{\max} \leq c_2 + \sqrt{\delta} \left[\sqrt{2\pi} \omega_{\max} + \sqrt{c_1} \right],$$

and for small δ we get $\omega_{\max} \leq 2c_2$.

Consider a fixed polygon. Let

$$T_{\Delta\theta} \triangleq A_{\Delta\theta}(0) \left/ \sum_i \left\{ g_i \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \right\} \right. \quad (12)$$

with $A_{\Delta\theta}(0)$ the area enclosed by the polygon at time zero. Recall that the denominator is $-\dot{A}_{\Delta\theta}$. Suppose that a side of the polygon disappears for $t < T_{\Delta\theta}$ and let \bar{T} be the first time that happens. If $0 \leq t \leq \bar{T}$ then $A_{\Delta\theta}(t)$ is bounded away from zero. Since the initial polygon is convex $\min_{0 \leq i \leq N-1} \omega_i(0)$ is positive. The estimates above imply that $\sup_{0 \leq t \leq \bar{T}} \max_{0 \leq i \leq N-1} \omega_i(t)$ is bounded. So $L_{\min}(\bar{T}) > 0$. This is a contradiction. Therefore the polygon will disappear in exactly $T_{\Delta\theta}$ units of time and no sides vanish before that.

3. CONVERGENCE OF THE CRYSTALLINE ALGORITHM FOR MOTION BY WEIGHTED CURVATURE

We wish to compare a convex polygon moving according to Eqs. (2) with a smooth simple closed convex curve moving according to Eq. (1). To do this we want to estimate the distance between the line containing the i th side of the polygon and the tangent to the curve parallel to this line. Since the (normal) velocities of these lines at time t are $\omega_i(t)$ and $\varpi_i(t) \triangleq \varpi(i\Delta\theta, t)$, respectively, we estimate the difference $\varpi_i - \omega_i$. All derivatives of ϖ remain bounded for $0 \leq t \leq \bar{T} < T$ where T is the time that the area enclosed by the curve vanishes (see Gage and Hamilton [2]). So, for these values of t , $\varpi_{\theta\theta\theta}$ is bounded and

$$\varpi_{\theta\theta}(i\Delta\theta, t) = \frac{\varpi_{i+1} - 2\varpi_i + \varpi_{i-1}}{2(1 - \cos \Delta\theta)} + O[(\Delta\theta)^2].$$

Using this in Eq. (7) and subtracting Eqs. (8),

$$\begin{aligned}
\dot{\varpi}_i - \dot{\omega}_i &= h_i \omega_i^2 \frac{(\varpi_{i+1} - \omega_{i+1}) - 2(\varpi_i - \omega_i) + (\varpi_{i-1} - \omega_{i-1})}{2(1 - \cos \Delta\theta)} \\
&\quad + h_i (\varpi_i - \omega_i) (\varpi_i + \omega_i) \frac{\varpi_{i+1} - 2\varpi_i + \varpi_{i-1}}{2(1 - \cos \Delta\theta)} \\
&\quad + h_i (\varpi_i - \omega_i) (\varpi_i^2 + \varpi_i \omega_i + \omega_i^2) + O[(\Delta\theta)^2].
\end{aligned}$$

Let $\Lambda_i \triangleq \varpi_i - \omega_i$. The last equation can be written

$$\dot{\Lambda}_i = a_i \frac{\Lambda_{i+1} - 2\Lambda_i + \Lambda_{i-1}}{2(1 - \cos \Delta\theta)} + b_i \Lambda_i + O[(\Delta\theta)^2] \quad (13)$$

with a_i positive, and b_i uniformly bounded if ω_{\max} is uniformly bounded. Therefore

$$\begin{aligned} \dot{\Lambda}_{\max} &\leq b\Lambda_{\max} + O[(\Delta\theta)^2] && \text{for } \Lambda_{\max} > 0, \\ \dot{\Lambda}_{\min} &\geq b\Lambda_{\min} + O[(\Delta\theta)^2] && \text{for } \Lambda_{\min} < 0, \end{aligned}$$

for $b = \sup_{0 \leq t \leq \tilde{T}} \max_i |b_i|$. So,

$$|\Lambda|_{\max}(t) \leq |\Lambda|_{\max}(0)e^{bt} + O[(\Delta\theta)^2]te^{bt} \quad (14)$$

Two important tasks remain to be done. One is to specify how we obtain the initial polygon and derive estimates for the curvature at time zero. The other one is to prove that if $\varpi_i - \omega_i$ is small then the curves are in fact uniformly close to one another. We address these issues in sequence.

(15 $_{\Delta\theta}$)

The initial polygon is obtained by the union of segments on lines tangent to the initial curve at the points with exterior normal $(\cos i\Delta\theta, \sin i\Delta\theta)$.

It should be emphasised that there should be a segment of nonzero length corresponding to each i . We derive estimates for $|\Lambda|_{\max}(0)$ and $|\Upsilon|_{\max}(0)$,

$$\Upsilon_i(0) \triangleq \varpi_\theta(i\Delta\theta, 0) - \frac{\omega_{i+1}(0) - \omega_i(0)}{\sin \Delta\theta}.$$

We need an estimate for $|\Upsilon|_{\max}$ to prove that c_2 (the constant in Eq. (11)) remains bounded as $\Delta\theta \rightarrow 0$. We prove that $k_i(0) = \kappa_i(0) + O[(\Delta\theta)^2]$ and $(k_{i+1}(0) - k_i(0))/\sin \Delta\theta = \kappa_\theta(i\Delta\theta, 0) + O(\Delta\theta)$ ($\kappa_i(t) \triangleq \kappa(i\Delta\theta, t)$). Since

$$\Lambda_i(0) = \varpi_i(0) - \omega_i(0) = g_i(\kappa_i(0) - k_i(0)) \quad (16)$$

and

$$\begin{aligned} \Upsilon_i(0) &= g_i \kappa_\theta(i\Delta\theta, 0) + g_\theta(i\Delta\theta) \kappa_i(0) \\ &\quad - g_{i+1} \frac{k_{i+1}(0) - k_i(0)}{\sin \Delta\theta} - \frac{g_{i+1} - g_i}{\sin \Delta\theta} k_i(0), \end{aligned} \quad (17)$$

it will follow that $|\Lambda|_{\max}(0) = O[(\Delta\theta)^2]$ and $|\Upsilon|_{\max}(0) = O(\Delta\theta)$. To estimate $k_i(0)$ we estimate the length $L_i(0)$. We decompose $L_i(0)$ into $L_i(0) = L_i^+ + L_i^-$ (see Fig. 3).

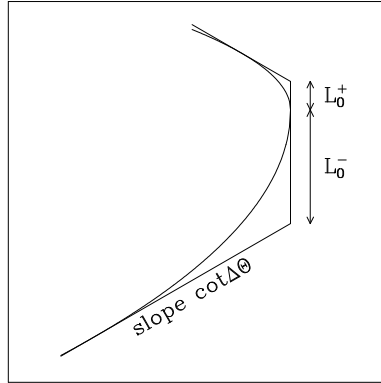


FIG. 3. The i th and part of the $(i-1)$ th and $(i+1)$ th sides at time zero. $L_i(0) = L_i^+ + L_i^-$. Here $N = 6$ and $i = 0$.

Note that

$$\kappa(\theta, 0) = \kappa_i(0) + (\theta - i\Delta\theta)\kappa_\theta(i\Delta\theta, 0) + O[(\theta - i\Delta\theta)^2]$$

and

$$\begin{aligned} r(\theta, 0) &= r(i\Delta\theta, 0) + \int_{i\Delta\theta}^{\theta} \frac{T(\vartheta)}{\kappa(\vartheta, 0)} d\vartheta \\ &= r(i\Delta\theta, 0) + \int_{i\Delta\theta}^{\theta} \frac{(-\sin \vartheta, \cos \vartheta)}{\kappa_i(0) + (\vartheta - i\Delta\theta)\kappa_\theta(i\Delta\theta, 0) + O[(\vartheta - i\Delta\theta)^2]} d\vartheta. \end{aligned}$$

Therefore

$$L_i^+ = \frac{1}{\kappa_i(0)} \int_0^{\Delta\theta} \frac{\cos \tau}{1 + \tau\rho_i + O(\tau^2)} d\tau - \cot \Delta\theta \frac{1}{\kappa_i(0)} \int_0^{\Delta\theta} \frac{\sin \tau}{1 + \tau\rho_i + O(\tau^2)} d\tau.$$

where $\rho_i = \kappa_\theta(i\Delta\theta, 0)/\kappa_i(0)$. One easily checks that

$$\begin{aligned} \cot \Delta\theta &= \frac{1}{\Delta\theta} - \frac{\Delta\theta}{3} + O[(\Delta\theta)^3], \\ \int_0^{\Delta\theta} \frac{\cos \tau}{1 + \tau\rho_i + O(\tau^2)} d\tau &= \Delta\theta - \frac{(\Delta\theta)^2}{2} \rho_i + O[(\Delta\theta)^3], \\ \int_0^{\Delta\theta} \frac{\sin \tau}{1 + \tau\rho_i + O(\tau^2)} d\tau &= \frac{(\Delta\theta)^2}{2} - \frac{(\Delta\theta)^3}{3} \rho_i + O[(\Delta\theta)^4]. \end{aligned} \tag{18}$$

So

$$L_i^+ = \frac{1}{\kappa_i(0)} \left[\frac{\Delta\theta}{2} - \frac{(\Delta\theta)^2}{6} \rho_i + O[(\Delta\theta)^3] \right]. \tag{19}$$

But then, of course,

$$L_i^- = \frac{1}{\kappa_i(0)} \left[\frac{\Delta\theta}{2} + \frac{(\Delta\theta)^2}{6} \rho_i + O[(\Delta\theta)^3] \right].$$

Adding, the expression for $L_i(0)$ in terms of $\kappa_i(0)$ and $\Delta\theta$ is

$$L_i(0) = \frac{1}{\kappa_i(0)} [\Delta\theta + O[(\Delta\theta)^3]].$$

This leads to the desired expression of $k_i(0)$ in terms of $\kappa_i(0)$,

$$\begin{aligned} k_i(0) &= \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \frac{1}{L_i(0)} \\ &= \frac{\Delta\theta + O[(\Delta\theta)^3]}{\frac{1}{\kappa_i(0)} [\Delta\theta + O[(\Delta\theta)^3]]} \\ &= \kappa_i(0) + O[(\Delta\theta)^2], \end{aligned} \tag{20}$$

and $(k_{i+1}(0) - k_i(0))/\sin \Delta\theta$ in terms of $\kappa_\theta(i\Delta\theta, 0)$,

$$\begin{aligned} \frac{k_{i+1}(0) - k_i(0)}{\sin \Delta\theta} &= \frac{\{\kappa_{i+1}(0) + O[(\Delta\theta)^2]\} - \{\kappa_i(0) + O[(\Delta\theta)^2]\}}{\Delta\theta + O[(\Delta\theta)^3]} \\ &= \frac{\kappa_{i+1}(0) - \kappa_i(0)}{\Delta\theta} + O(\Delta\theta) \\ &= \kappa_\theta(i\Delta\theta, 0) + O(\Delta\theta). \end{aligned} \tag{21}$$

Now we prove that the curves are uniformly close to one another. It is convenient to use the Hausdorff metric on compact sets A and B , defined as

$$D(A, B) \triangleq \max \left\{ \max_{p \in A} \min_{q \in B} \text{dist}(p, q), \max_{q \in B} \min_{p \in A} \text{dist}(p, q) \right\}.$$

Thus, A and B are close if any point of A is close to B and vice-versa. Our main result is the following

THEOREM. *Suppose f (the interfacial energy) is positive, C^3 , and $f + f'' > 0$. Consider a smooth simple closed convex curve $C(t)$ enclosing at time zero an area A and moving according to Eq. (1). Consider further a polygon $P(t)$ determined initially by condition $(15_{\Delta\theta})$ and moving according to Eqs. (2). Let $T \triangleq A / \int_0^{2\pi} (f(\theta) + f''(\theta)) d\theta$ and choose any $\tilde{T} < T$ (T is the time it takes for the curve to shrink to a point or to a straight line). If $0 \leq t \leq \tilde{T}$ and $\Delta\theta$ is sufficiently small then $P(t)$ is close to $C(t)$ in the sense that $D(P(t), C(t)) = O[(\Delta\theta)^2]$.*

Proof. By Eqs. (16), (17), (20), and (21), condition $(15_{\Delta\theta})$ implies that

$$|\Lambda|_{\max}(0) = \max_{0 \leq i \leq N-1} |\varpi(i\Delta\theta, 0) - \omega_i(0)| = O[(\Delta\theta)^2]$$

and

$$|\Upsilon|_{\max}(0) = \max_{0 \leq i \leq N-1} \left| \varpi_\theta(i\Delta\theta, 0) - \frac{\omega_{i+1}(0) - \omega_i(0)}{\sin \Delta\theta} \right| = O(\Delta\theta).$$

These estimates imply that $\omega_{\min}(0)$ is bounded away from zero, $\omega_{\max}(0)$ is bounded, and $(\omega_{i+1}(0) - \omega_i(0))^2 / [2(1 - \cos \Delta\theta)]$ is bounded with these bounds uniform in $\Delta\theta$. Pick $\Delta\theta$ sufficiently small so that $T_{\Delta\theta} > \tilde{T} + \delta$ for some positive δ and $A_{\Delta\theta}(\tilde{T})$ is bounded away from zero uniformly in $\Delta\theta$ ($T_{\Delta\theta}$ is defined in Eq. (12)). Then the bounds of Section 2 are uniform in $\Delta\theta$. Since $0 \leq t \leq \tilde{T} < T$, $\varpi_{\theta\theta\theta}$ is uniformly bounded. Then Eq. (14) implies

$$|\varpi(i\Delta\theta, t) - \omega_i(t)| \leq c_3(\Delta\theta)^2. \quad (22)$$

We define $C_P(t)$ to be the polygon obtained by the union of segments on lines tangent to $C(t)$ at points with exterior normal $-N_i$. We prove that $D(P, C) = O[(\Delta\theta)^2]$ in two steps. First we estimate $D(P, C_P)$ and then we estimate $D(C, C_P)$.

We claim that the distance $D(P, C_P) = O[(\Delta\theta)^2]$. The (normal) velocity of the line containing the i th side of the polygon P is ω_i while the (normal) velocity of the tangent to the curve parallel to this line is ϖ_i . Since these lines coincide at time zero inequality (22) implies that their distance at time t is less or equal to $c_3 t (\Delta\theta)^2$. Furthermore, this is true for all i so P lies in a strip of width $2c_3 t (\Delta\theta)^2$ whose center is C_P . The Hausdorff distance between C_P and the boundary of this strip is $c_3 t (\Delta\theta)^2 / \cos(\Delta\theta/2)$. Hence the distance $D(P, C_P) = O[(\Delta\theta)^2]$.

Now we claim that the distance $D(C, C_P) = O[(\Delta\theta)^2]$. This distance is equal to the distance between some vertex of C_P and the point on C closest to it, because C is convex. If the vertex in question is the $(i+1)$ th one, then $D(C, C_P)$ is less than the distance between the $(i+1)$ th vertex of C_P and the point on C with exterior normal $(\cos(i+1/2)\Delta\theta, \sin(i+1/2)\Delta\theta)$. The point on C with exterior normal $(\cos(i+1/2)\Delta\theta, \sin(i+1/2)\Delta\theta)$ is

$$p_i + \int_{i\Delta\theta}^{(i+1/2)\Delta\theta} \frac{T(\vartheta)}{\kappa(\vartheta, t)} d\vartheta,$$

where p_i is the point of tangency between C and the side of C_P with exterior normal $-N_i$, or

$$p_i + \frac{1}{\kappa_i} \int_0^{\Delta\theta/2} \frac{\cos \tau}{1 + \tau \rho_i + O(\tau^2)} d\tau T_i + \frac{1}{\kappa_i} \int_0^{\Delta\theta/2} \frac{\sin \tau}{1 + \tau \rho_i + O(\tau^2)} d\tau N_i;$$

by Eqs. (18) this equals

$$p_i + \left[\frac{\Delta\theta}{2\kappa_i} + O[(\Delta\theta)^2] \right] T_i + O[(\Delta\theta)^2] N_i.$$

On the other hand, by Eqs. (19) the $(i+1)$ th vertex of C_P is

$$p_i + \left[\frac{\Delta\theta}{2\kappa_i} + O[(\Delta\theta)^2] \right] T_i.$$

We conclude that the distance between the $(i+1)$ th vertex of C_P and the point on C with normal $(\cos(i+1/2)\Delta\theta, \sin(i+1/2)\Delta\theta)$ is $O[(\Delta\theta)^2]$.

Finally,

$$D(P, C) \leq D(P, C_P) + D(C, C_P) = O[(\Delta\theta)^2].$$

□

We close with two remarks. The first one concerns the comparison between ϖ and its discrete analogue. Above we proved a result of convergence in the sup norm. It is also possible to prove that

$$\left| \varpi_\theta(i\Delta\theta, t) - \frac{\omega_{i+1}(t) - \omega_i(t)}{\sin \Delta\theta} \right| = O(\Delta\theta),$$

for $0 \leq t \leq \tilde{T}$. The proof goes in three steps: (i) one derives the evolution equation for $(\omega_{i+1} - \omega_i)/\sin \Delta\theta$ using Eq. (8), (ii) one differentiates Eq. (7) with respect to θ and writes an analogue of the equation obtained in the previous step by substituting derivatives of ϖ_θ by difference quotients, and (iii) one subtracts the equations obtained in the previous two steps, rewrites the difference in an appropriate way (as in Eq. (13)), and derives a differential inequality for the maximum of the absolute value of the difference we want to estimate. One estimates $(\omega_{i+1} - \omega_i)/\sin \Delta\theta$ and not $(\omega_{i+1} - \omega_{i-1})/(2\sin \Delta\theta)$ directly to be able to make conclusions about the sign of the difference quotients that appear in step three.

Our final remark concerns the comparison between the order of convergence of this numerical scheme for simple closed convex curves, on the one hand, and for graphs with prescribed boundary conditions, on the other. The main result of this paper is that the former order of convergence is $O[(\Delta\theta)^2]$ in the Hausdorff norm, when the admissible angles are equally spaced. The latter one is $O(\sqrt{\Delta\theta})$ in the stronger norm H^1 , for a general distribution of admissible slopes (see [3]).

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